Euler-Lagrange equations
Monday, June 29th 2015

Solutions Problem 1
Since the functional that describes the length of a $\mathcal{C}^1$ curve parametrized by $x(t),\ t \in [a,b]$, is

$$J(x(t)) = \int_a^b \sqrt{1 + (\dot{x}(t))^2} dt,$$

the corresponding Lagrangian function is

$$L(t, x, \dot{x}) = \sqrt{1 + (\dot{x})^2}.$$

Notice that this Lagrangian function does not depend explicitely on x. Therefore, the Euler-Lagrange equation writes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) = \frac{d}{dt}\frac{\dot{x}(t)}{\sqrt{1 + (\dot{x}(t))^2}} = 0$$

which means that the function $t \to \frac{\dot{x}(t)}{\sqrt{1+(\dot{x}(t))^2}}$ is constant on $[a,b]$. Denote $c \in \mathbb{R}$ this constant value. For all $t \in [a,b]$, we have

$$\dot{x}(t) = c\sqrt{1 + (\dot{x}(t))^2}$$

so, if $c = 1$, then

$$\dot{x}(t) = 0$$

and, if $c \neq 1$, then

$$\dot{x}(t) = \Big(\frac{c^2}{1 - c^2}\Big)^{\frac{1}{2}}.$$

Hence, $\dot{x}(t)$ is constant on $[a,b]$ so $x(t)$ is a straight line defined on $[a,b]$.

Solutions Problem 2
Let $x(t),\ t \in [a,b]$, be a minimizer of the problem. For any curve $\bar{x}(t),\ t \in [a,b]$, such that $\bar{x}(a) = x_0$ and $\bar{x}(b)$ is free, define the corresponding perturbation $h(t) = \bar{x}(t) - x(t),\ t \in [a,b]$. This perturbation satisfies $h(a) = 0$ but

$h(b)$ is free. Therefore, the fundamental formula of the calculus of variations writes

$$\begin{aligned}\Delta C(h) &= \int_a^b \Big(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\Big) h(t)dt \\ &\quad + \Big[\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))h(t)\Big]_a^b + \Big[L(t, x(t), \dot{x}(t))\delta t\Big]_a^b \\ &= \int_a^b \Big(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\Big) h(t)dt \\ &\quad + \frac{\partial L}{\partial \dot{x}}(b, x(b), \dot{x}(b))h(b).\end{aligned}$$

The curve x being a minimizer, we must have

$$\Delta C(h) \geq 0 \tag{1}$$

for each h and, by linearity of ΔC, we get

$$\Delta C(h) = 0 \tag{2}$$

for each h. In particular, this conditions holds for every perturbation h satisfying $h(b) = 0$ for which, combining equations (1) and (3), we get

$$\int_a^b \Big(\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t))\Big) h(t)dt = 0.$$

This means that the standard Euler-Lagrange equation

$$\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)),\ t \in [a, b],$$

still holds. As a result, for every perturbation h, the integral involved in equation (1) vanishes and we get

$$\frac{\partial L}{\partial \dot{x}}(b, x(b), \dot{x}(b))h(b) = 0.$$

Since $h(b)$ is arbitrary, this leads to the additional optimality condition

$$\frac{\partial L}{\partial \dot{x}}(b, x(b), \dot{x}(b)) = 0$$

Solutions Problem 3

Let $x(t),\ t \in [a, t_f]$, be a minimizer of the problem. For any curve $\bar{x}(t),\ t \in [a, t_f + \delta t_f]$, such that $\bar{x}(a) = x_0$ and $\bar{x}(t_f + \delta t_f) = \phi(t_f + \delta t_f)$, define the corresponding perturbation $h(t) = \bar{x}(t) - x(t),\ t \in [a, \max\{t_f, t_f + \delta t_f\}]$

(either x or $\bar{x}$ can be considered 0 if not well defined). The fundamental formula of the calculus of variations writes

$$\begin{aligned}
\Delta C(h) = &\int_a^{t_f} \Big(\frac{\partial L}{\partial x}(t,x(t),\dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t,x(t),\dot{x}(t))\Big)h(t)dt \\
&+ \Big[\frac{\partial L}{\partial \dot{x}}(t,x(t),\dot{x}(t))\delta x(t)\Big]_a^{t_f} \\
&+ \Big[\Big(L(t,x(t),\dot{x}(t)) - \dot{x}(t)\frac{\partial L}{\partial \dot{x}}(t,x(t),\dot{x}(t))\Big)\delta t\Big]_a^{t_f} \\
= &\int_a^{t_f} \Big(\frac{\partial L}{\partial x}(t,x(t),\dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t,x(t),\dot{x}(t))\Big)h(t)dt \qquad (3) \\
&+ \Big[\frac{\partial L}{\partial \dot{x}}(t,x(t),\dot{x}(t))\delta x(t)\Big]_a^{t_f} \\
&+ \Big(L(t_f,x(t_f),\dot{x}(t_f)) - \dot{x}(t_f)\frac{\partial L}{\partial \dot{x}}(t_f,x(t_f),\dot{x}(t_f))\Big)\delta t_f
\end{aligned}$$

Using a similar argument as in problem 2, we have

$$\Delta C(h) = 0 \qquad (4)$$

for each h since x is a minimizer. Considering a perturbation h preserving the final time ($\delta t_f = 0$ and so $\delta x(t_f) = 0$ since $\bar{x}(t_f) = \phi(t_f) = x(t_f)$), we see that the standard Euler-Lagrange equation still holds. As in problem 2, the integral involved in equation (4) vanishes for every perturbation h. Furthermore, using the fact that $\delta x(a) = 0$ (always true since $x(a) = x_0$) and combining equations (3) and (4), we get

$$\frac{\partial L}{\partial \dot{x}}(t_f,x(t_f),\dot{x}(t_f))\Big(\delta x(t_f) - \dot{x}(t_f)\delta t_f\Big) + L(t_f,x(t_f),\dot{x}(t_f))\delta t_f = 0. \qquad (5)$$

But $x(t_f) = \phi(t_f)$ so $\frac{\delta x(t_f)}{\delta t_f} = \phi'(t_f)$. Plugging in (5), we get the additional optimality condition

$$\frac{\partial L}{\partial \dot{x}}(t_f,x(t_f),\dot{x}(t_f))\Big(\phi'(t_f) - \dot{x}(t_f)\Big) + L(t_f,x(t_f),\dot{x}(t_f)) = 0. \qquad (6)$$

which is called the transversality condition.